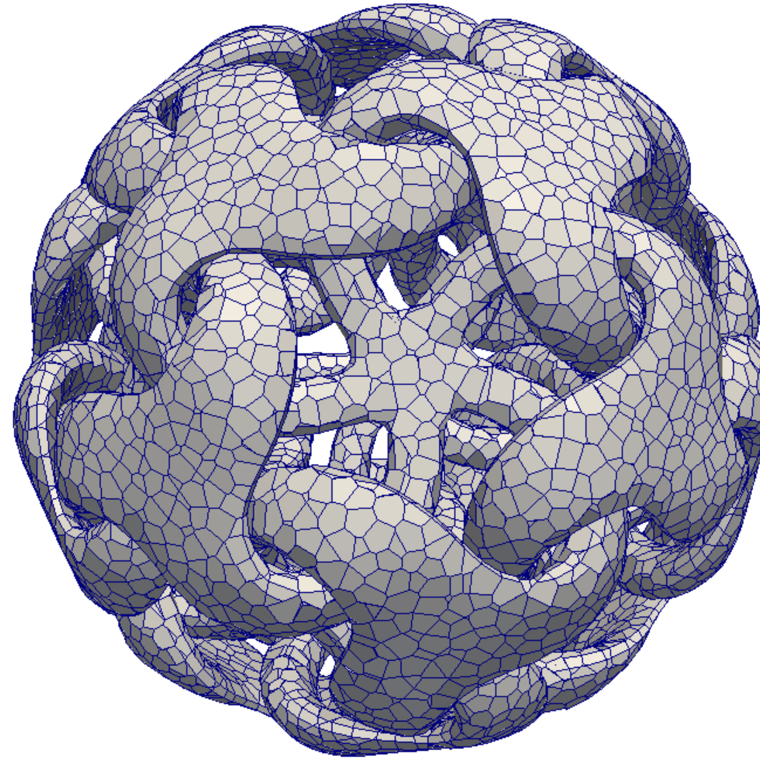
Notre Dame de Paris, Gargoyle, by FreiGurita (1968)

+two other 2011 papers



Uniform Random Voronoi Meshes

Mohamed S. Ebeida & **Scott A. Mitchell** (speaker)

20th International Meshing Roundtable
Paris, France

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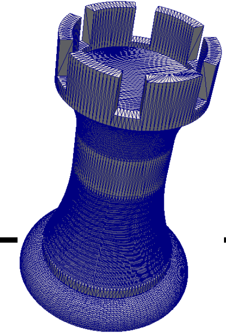
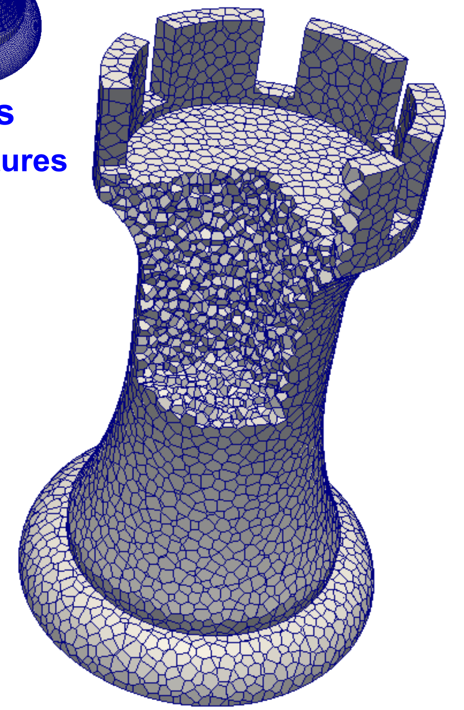
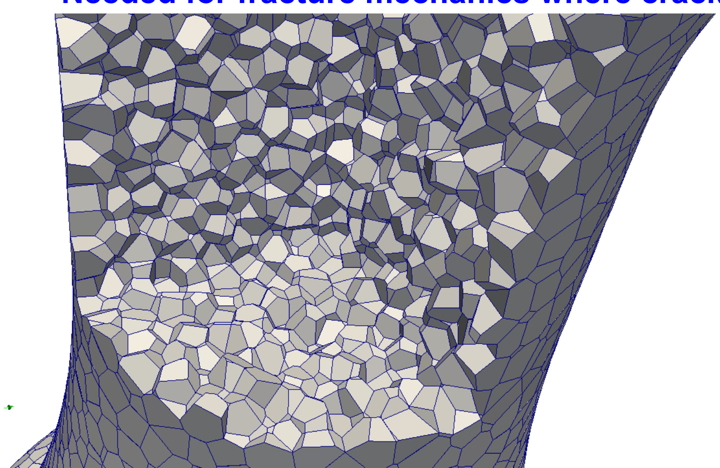
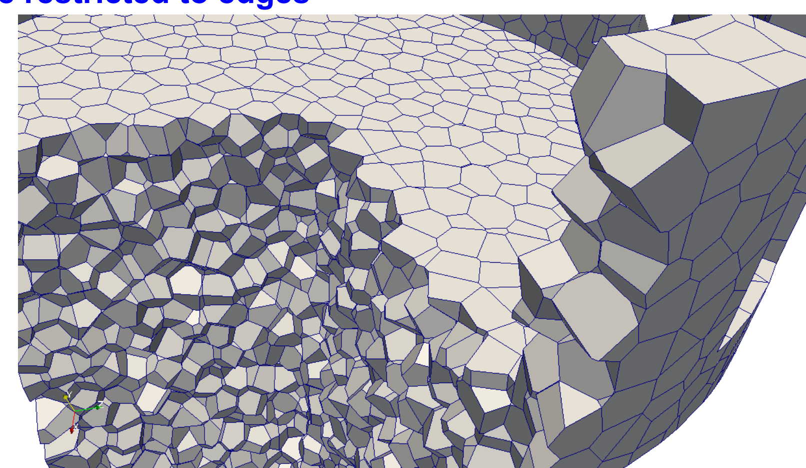
Summary





- Random Polyhedral Meshing
  - Generate random points using the maximal Poisson-disk process
    - Points placed on reflex boundary features, but not concave or flat features
    - Contrast to primal methods
  - Symbolically split points (not in paper)
  - Construct Voronoi cells
    - Bounding box, cut by boundary and Voronoi planes
      - Bounding box works because cells have bounded size
    - Small edges collapsed
- Get
  - Voronoi mesh of convex polyhedral cells
  - Bounded cell aspect ratio and facet dihedrals
  - Random orientation of mesh edges
    - Needed for fracture mechanics where cracks are restricted to edges





Maximal Poisson-Disk Sampling (MPS)

- **What is MPS?**

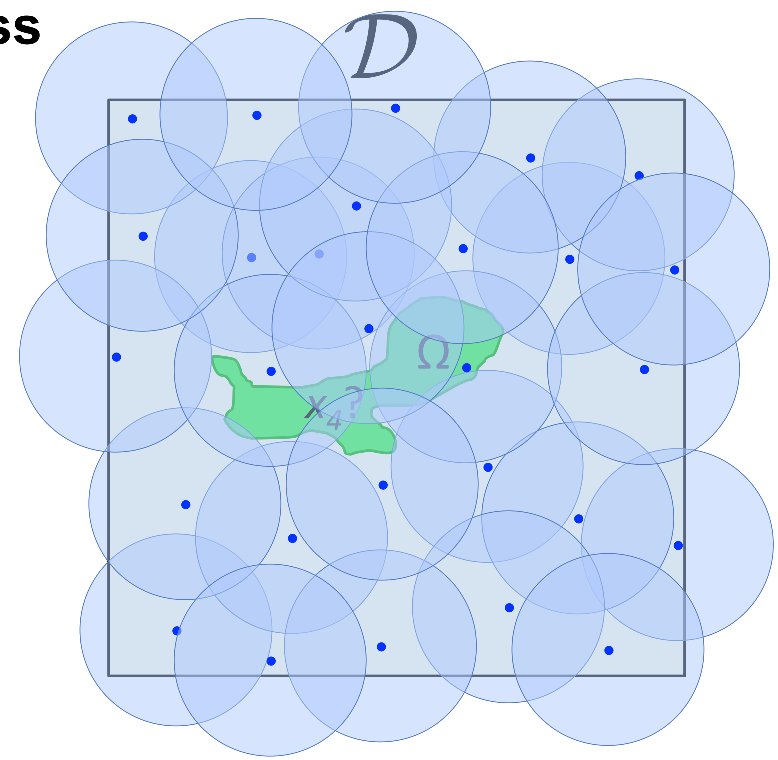
  - **Dart-throwing**
  - **Insert random points into a domain, build set X**
    - **With the "Poisson" process**

Empty disk: $\forall x_i, x_j \in X, x_i \neq x_j : ||x_i - x_j|| \geq r$

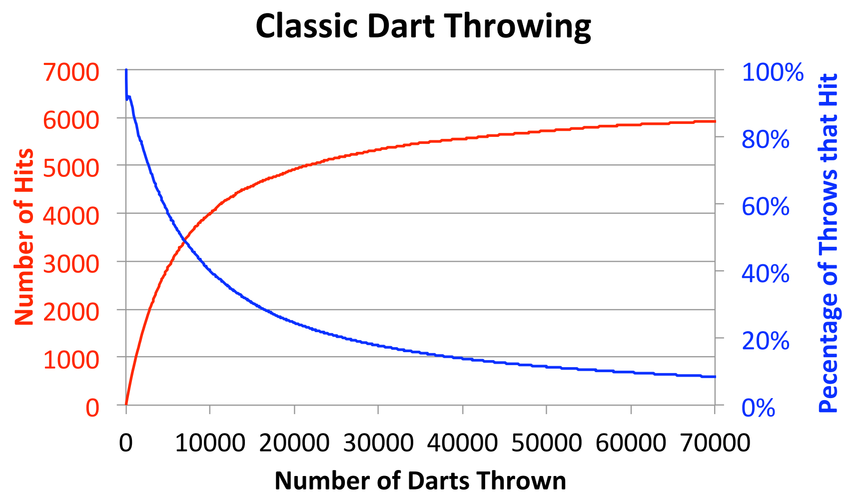
Bias-free: $\forall x_i \in X, \forall \Omega \subset \mathcal{D}_{i-1} :$

$$P\left(x_i \in \Omega\right) = \frac{\text{Area}(\Omega)}{\text{Area}(\mathcal{D}_{i-1})}$$

Maximal: $\forall x \in \mathcal{D}, \exists x_i \in X : ||x - x_i|| < r$

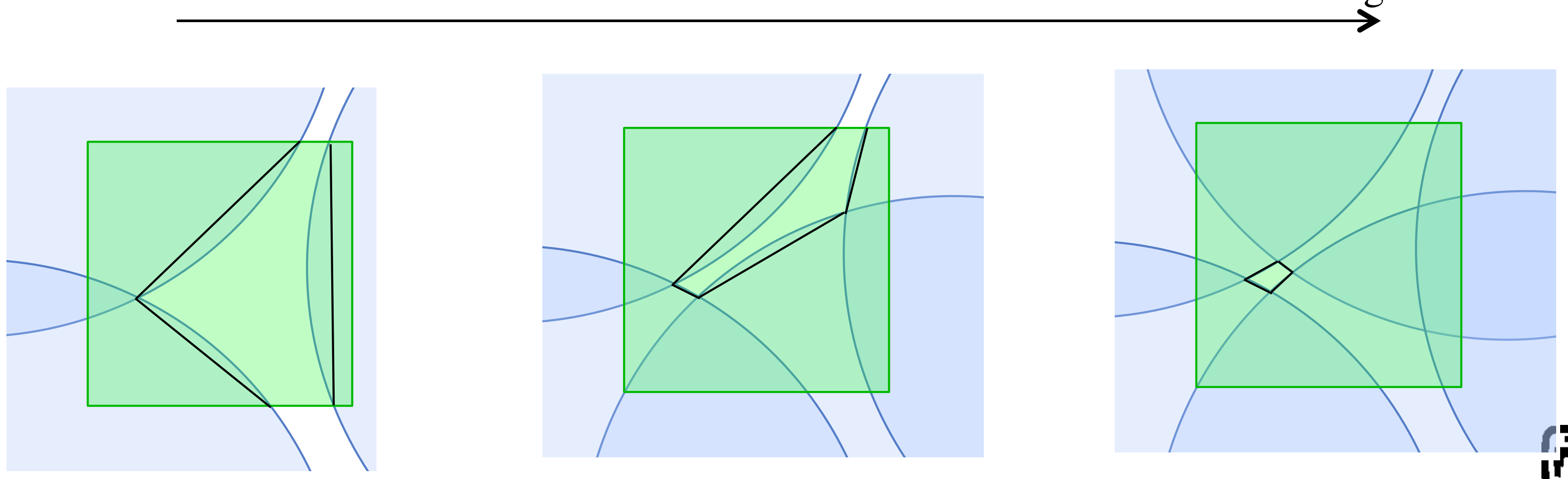


Statistical Process ≠ Algorithm



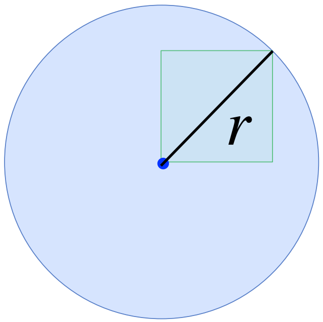
Algorithm progress

sliver regions

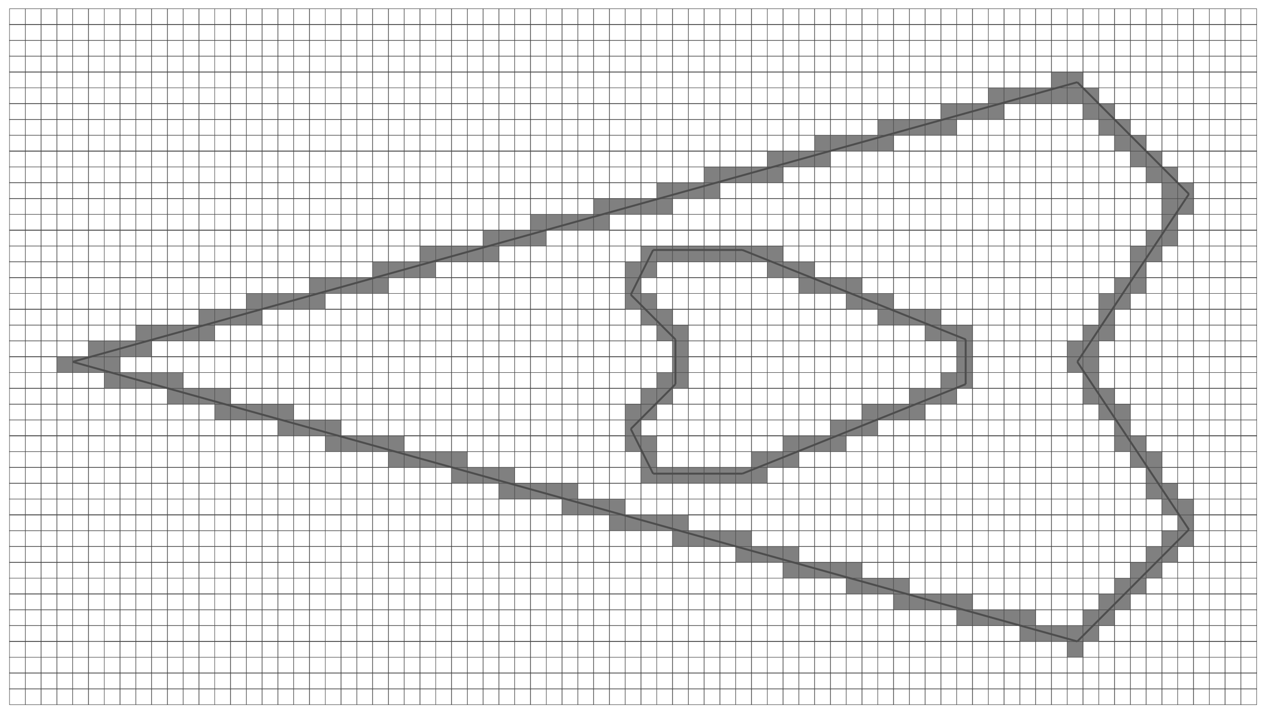


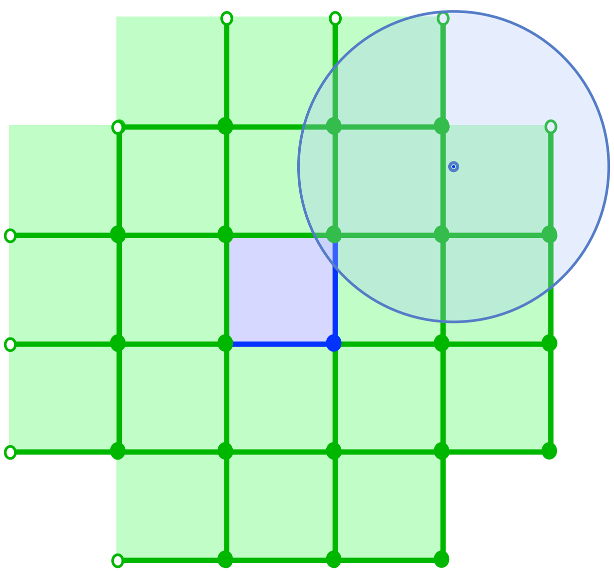
"Efficient maximal Poisson-disk sampling"

First provably correct, time- space-optimal algorithm.
Mohamed S. Ebeida, Anjul Patney, Scott A. Mitchell,
Andrew Davidson, Patrick M. Knupp, and John D. Owens.
ACM Transactions on Graphics (Proc. SIGGRAPH 2011), 30(4), 2011.



Background grid of squares (cubes…) for locality

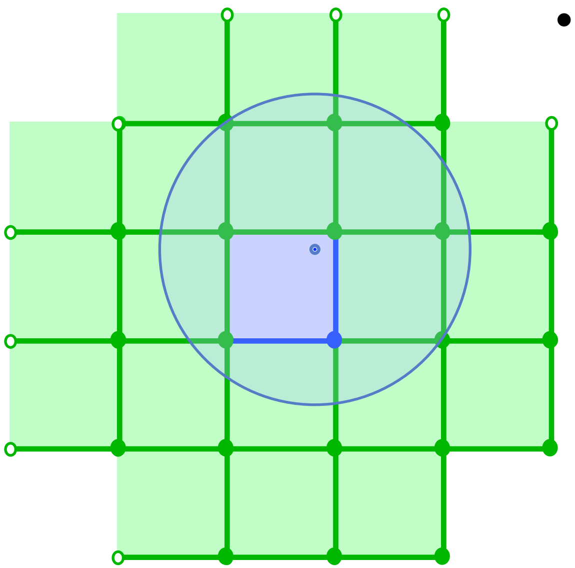




Sid Meier's Civilization Template

Everything is O(1)

Efficient maximal Poisson-disk sampling

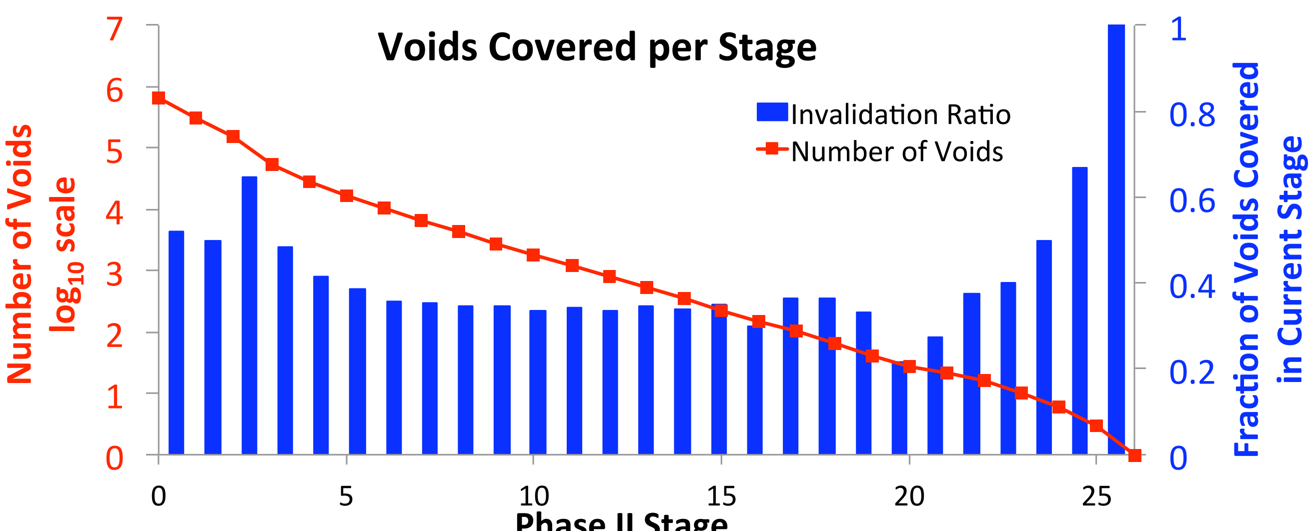


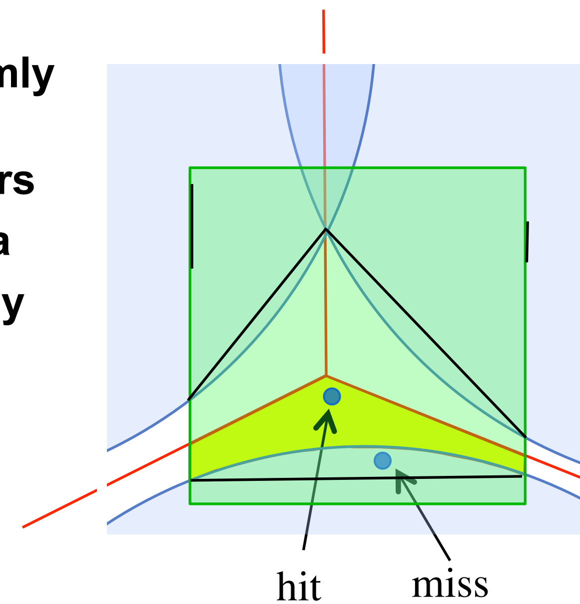
- **Algorithm**
  - **Phase I**
    Throw darts in squares
    - **Pick square uniformly**
    - **Pick point in square uniformly**
  - **Phase II**
    Throw darts in polygons $\supset$ slivers
    - **Pick sliver weighted by area**
    - **Pick point in sliver uniformly**

Bias-free: $\forall x_i \in X, \forall \Omega \subset \mathcal{D}_{i-1}$:

without selecting from entire domain

$$P\left(x_i \in \Omega\right) = \frac{\text{Area}(\Omega)}{\text{Area}(\mathcal{D}_{i-1})}$$





E(n) throws proof idea

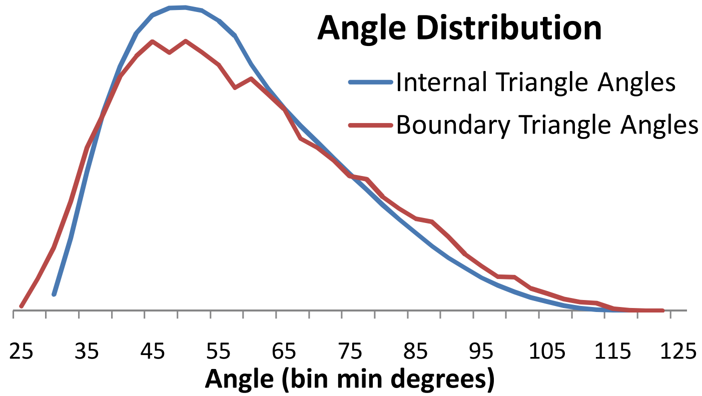
- **Hit/miss ratio = Voronoi cell area ratio > constant.**

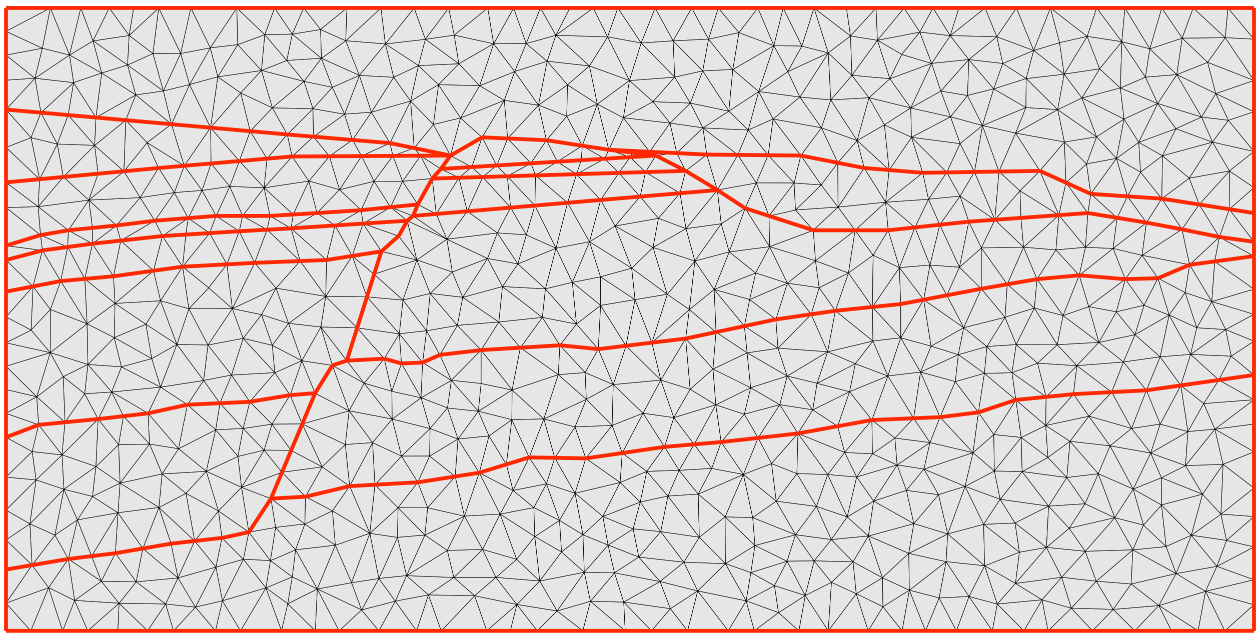
In practice, use flat implicit octree in d>2

Also Triangular Meshes

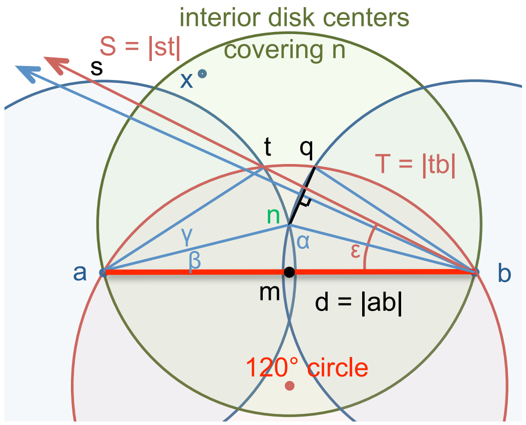
“Efficient and good Delaunay meshes from random points.”
Mohamed S. Ebeida, Scott A. Mitchell, Andrew A. Davidson, Anjul Patney, Patrick M. Knupp, and John D. Owens.
Computer-Aided Design, 2011. Proc. 2011 SIAM Conference on Geometric and Physical Modeling (GD/SPM11).

- **Reverse cause-effect**
  - **Delaunay Refinement:**
    Insert circle-centers to kill large Delaunay circles
    - **Maximal sample results**
  - **MPS: Insert points randomly to maximally sample**
    - **Small Delaunay circles result**
  - **Nearly identical angle bounds either way**
    - **Delaunay circle-centers can be ignored!**





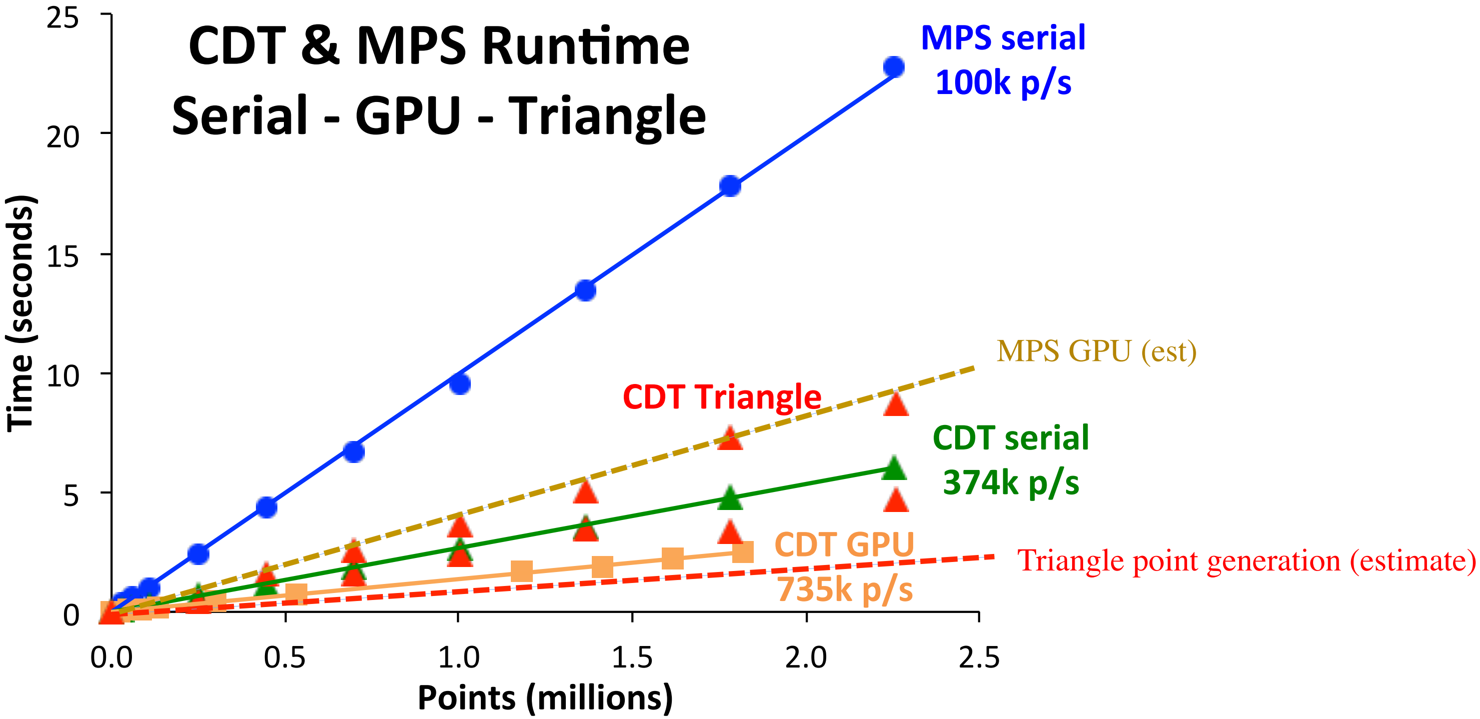
Cover the boundary with random disks



- **Simple algorithm for covering the boundary randomly**
  - **Complicated geometric proof**

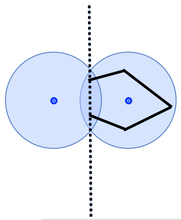
"Efficient" for MPS, scales great, but how fast?

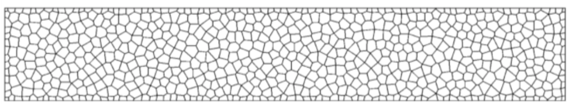
- **Delaunay refinement**
  - **Points from deterministic process - fast**
- **MPS**
  - **Points from strict unbiased random process – slow**
- **But once points are generated we're as fast as Triangle, and our GPU code is 2x faster**



What is MPS good for?

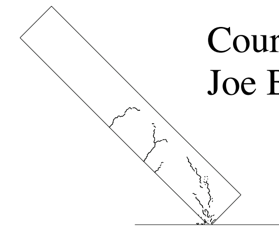
- **Fracture mechanics simulations**
  - **Fractures occur on Voronoi cell boundaries**
    - **Mesh variation ⊂ material strength variation**
      - **Ensembles of simulations**
    - **Unbiased sampling gives realistic cracks**
      - **Edge orientations are uniform random**
  - **Domains: non-convex, internal boundaries**

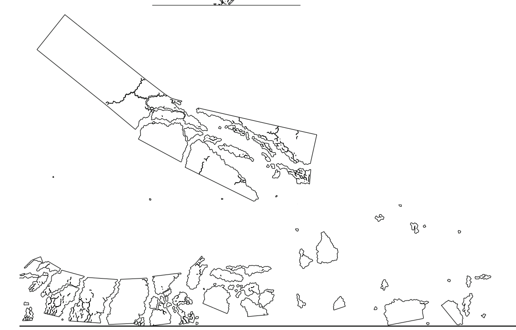




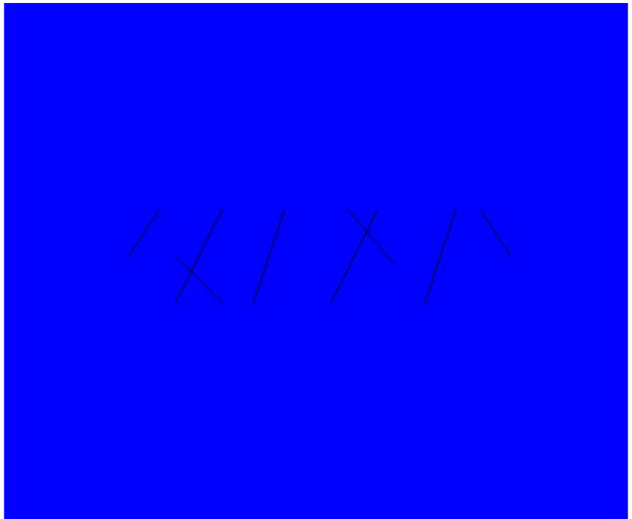
Fracture Simulations

Courtesy of
Joe Bishop (SNL)

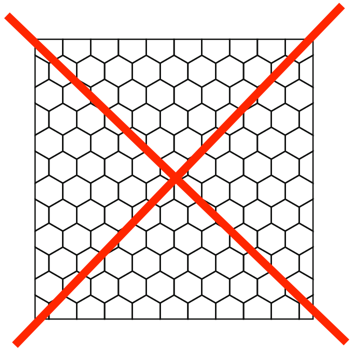




Impact
Joe Bishop, SNL org 1500
Fracture simulation
Need random meshes because
cracks are along edges



Alternatives



- Voronoi Mesher
  - CVT Centroidal Voronoi Tessellation
    - Seed = cell’s center of mass
    - Via iterative adjustment of seed location
    - Good shaped cells, but “biased”, regular mesh
    - Target app: fracture simulations with fracture along mesh edges
- Primal meshers
  - Miller: maximal disk packings for bounded edge-radius tet meshes
  - Shimada and Gossard Bubble meshes
    - Force network, insertion and removal

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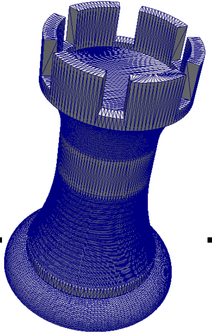
IMR paper algorithm!



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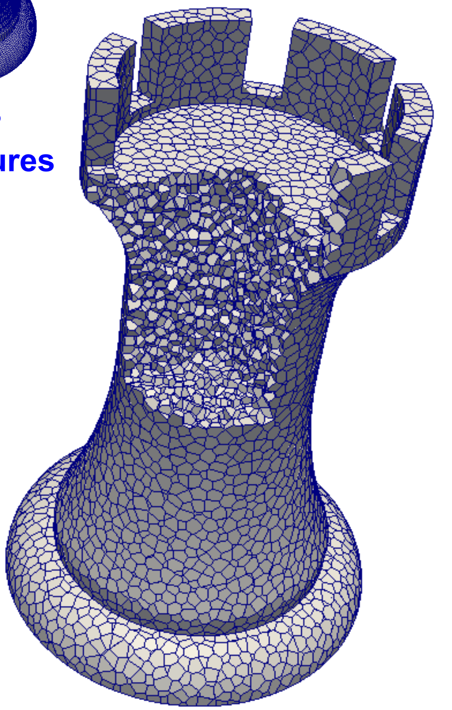
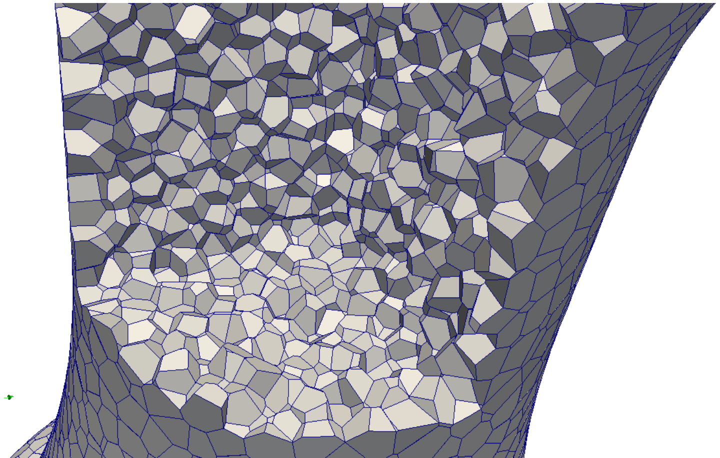
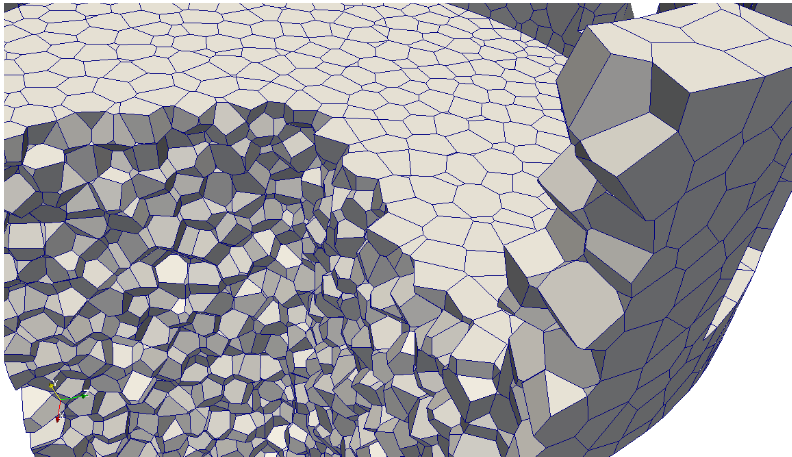
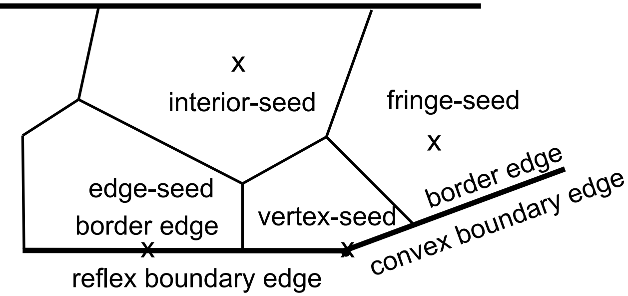




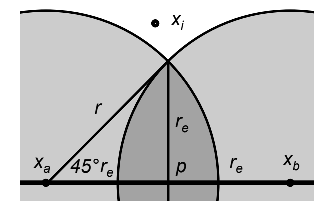


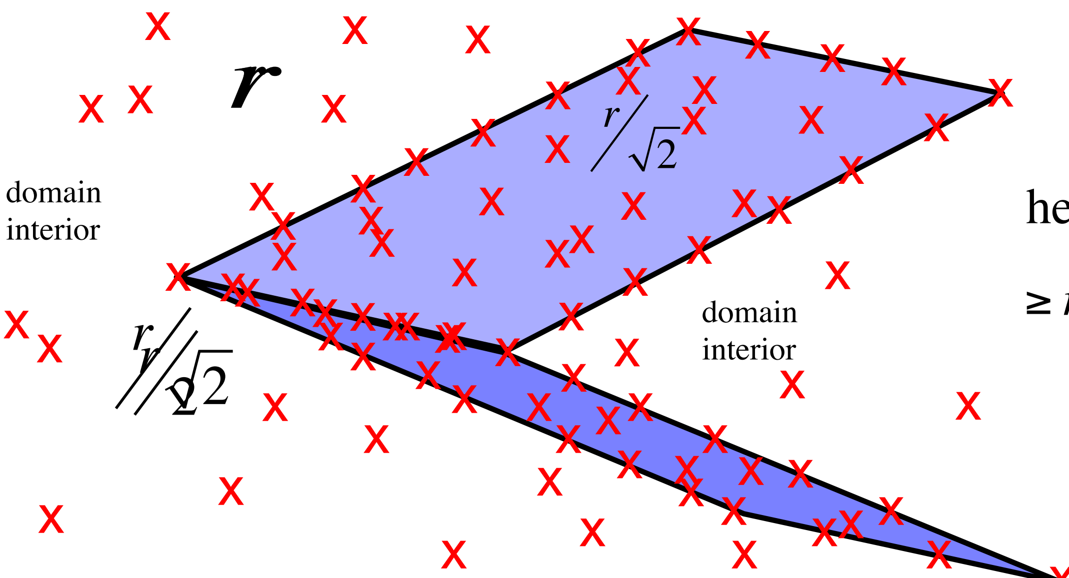
Boundary Sampling

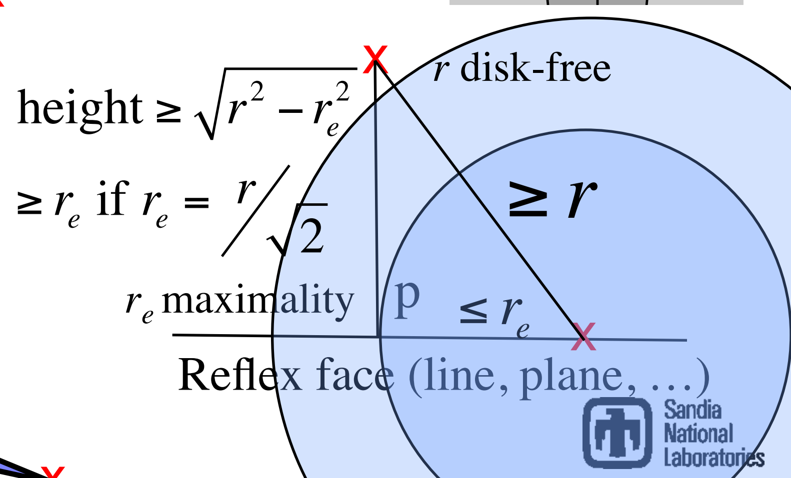


- **Maximally sample**
  - **Points interior to domain, not on boundary…**
    …unless we have to:
    - **Reflex features require special care, not sharp ones**
      - **"Reflex" includes 2-sided facets**
    - **Not the dual of a body-fitted primal mesh**
      - **Better (not constant 90°) dihedrals at boundary**
- **Goal: cells align with boundary features, cells are convex**
- **Sufficient: every point on a reflex face is closest to a sample from that reflex feature (or sub-facet)**
  - **Sqrt(2) denser sampling on reflex feature**

$$r_e = r/\sqrt{2}$$







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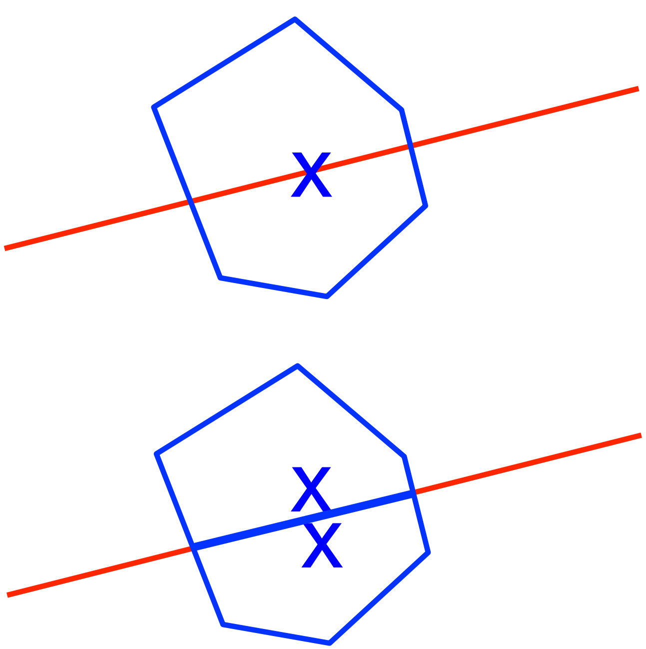
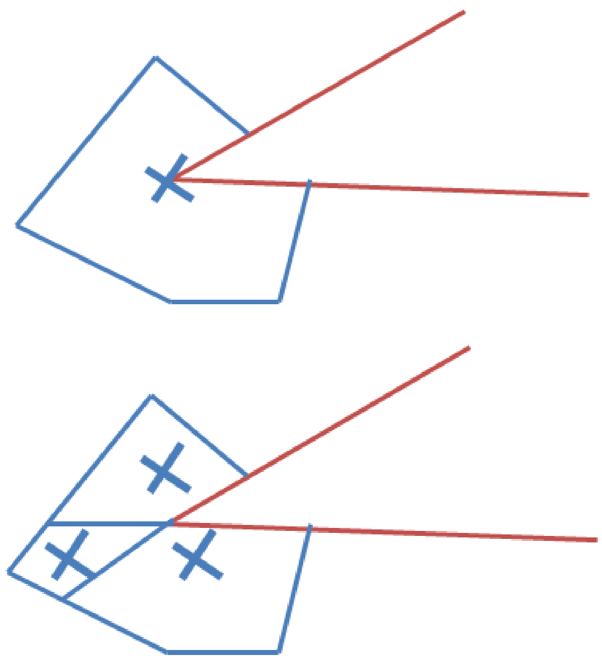
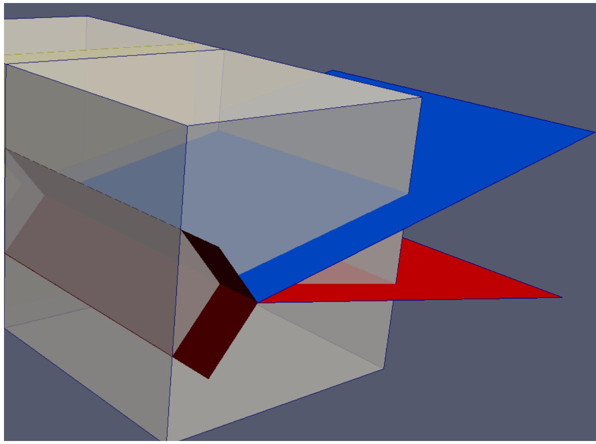
Bonus: Convex Cells
Paper: star-shaped cells at reflex faces

- **Clipping by boundary**
  - By prior page only non-reflex (convex) boundary features affect interior samples
  - Intersection of convex Voronoi cell w/ convex boundary = convex clipped cell
- **Symbolic duplication of reflex samples**







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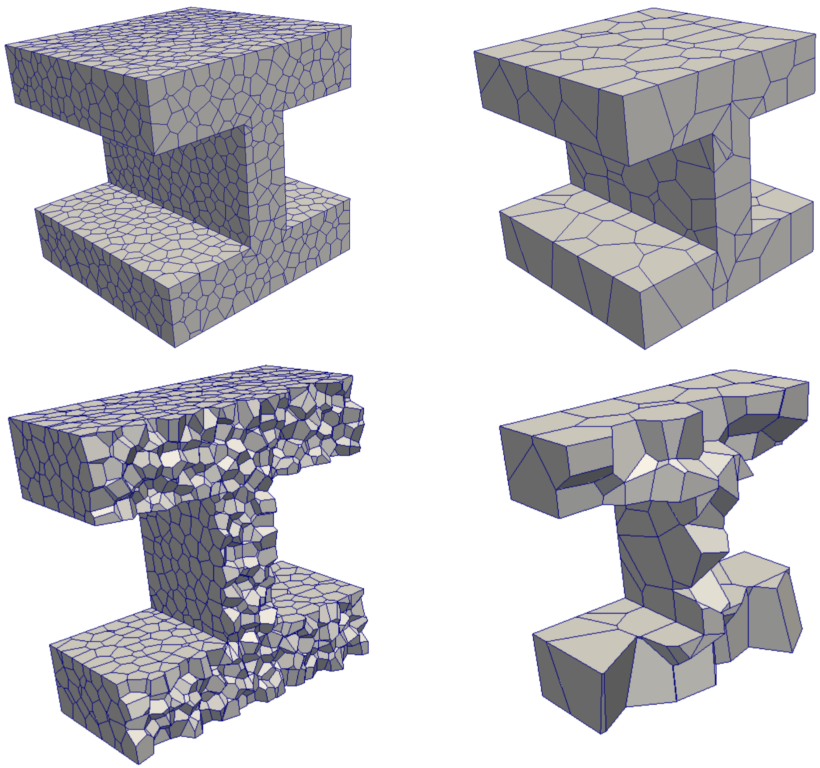
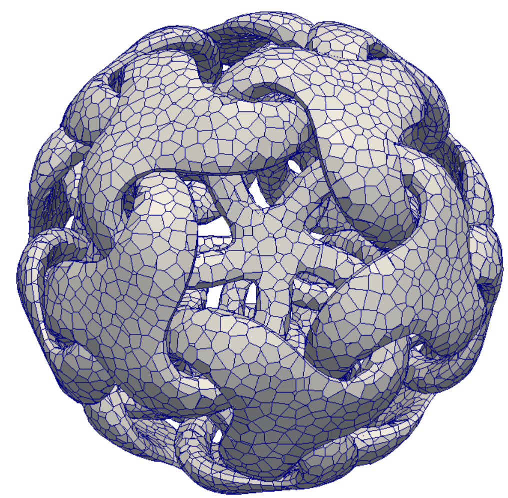
Voronoi Quality

- Provable facet dihedral angle bounds
- Provable cell aspect ratios





Quality proof idea

$$\cancel{\text{Bias free: } \forall \Omega \in \mathcal{D}_{i-1} : P(x_i \in \Omega) = \frac{\text{Area}(\Omega)}{\text{Area}(\mathcal{D}_{i-1})}} \quad \text{(1a)}$$

$$\text{Empty disk: } \forall x_i, x_j \in X, i \neq j : ||x_i - x_j|| \geq r \quad \text{(1b)}$$

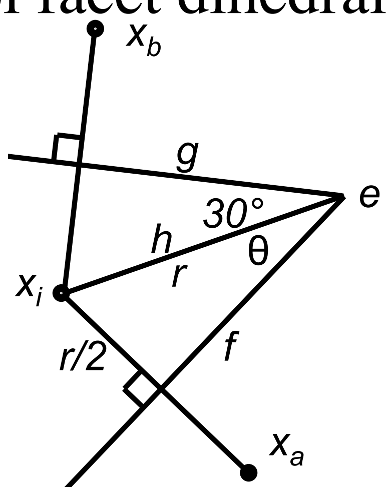
$$\text{Maximal: } \forall p \in \mathcal{D}, \exists x_i \in X : ||p - x_i|| < r \quad \text{(1c)}$$

Voronoi:

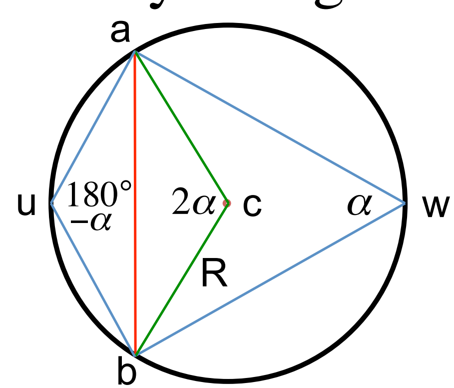
$$V_i = \{p\} \in \mathcal{D} : \forall j, ||p - x_i|| \leq ||p - x_j||$$

- **"Maximality" bounds the maximum distance from Voronoi cell seed to its vertices**
  = Delaunay vertex to circle center
- **"Disk-free" bounds the minimum distance between two seeds**
  = a Delaunay edge

Voronoi facet dihedral angles:



Delaunay triangle angles:



(b) Central Angle Theorem.

as Chew 89

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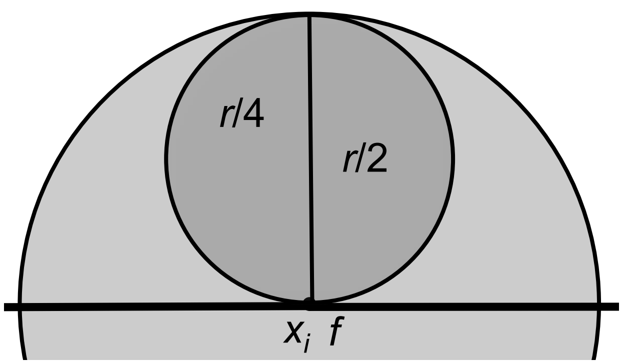
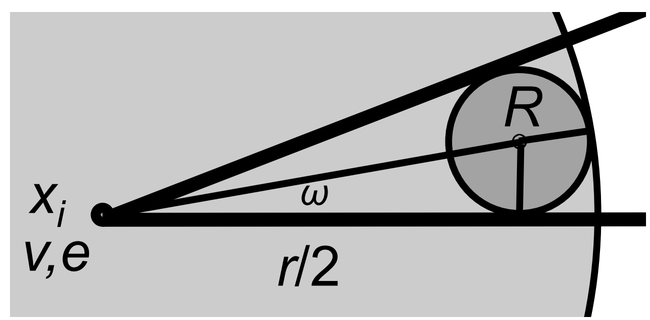
Aspect Ratio Proofs (star-shaped cells)

- **Aspect ratio**
  - **Circumscribed sphere radius < r (from maximality)**
  - **Inscribed sphere radius > some factor r (from disk-free)**

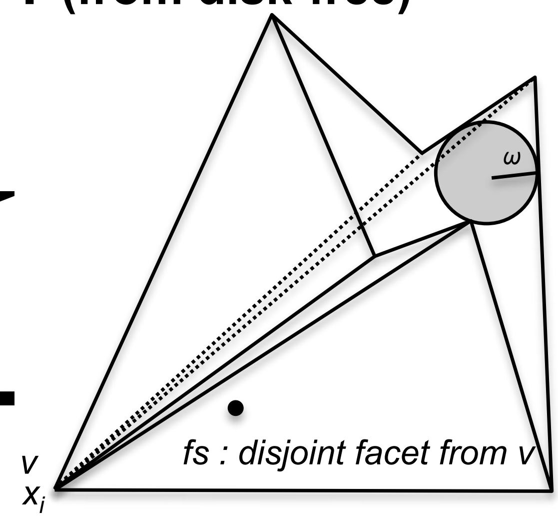
    If cell is interior: r/2



Clipped by one facet: r/4



Facets of one edge

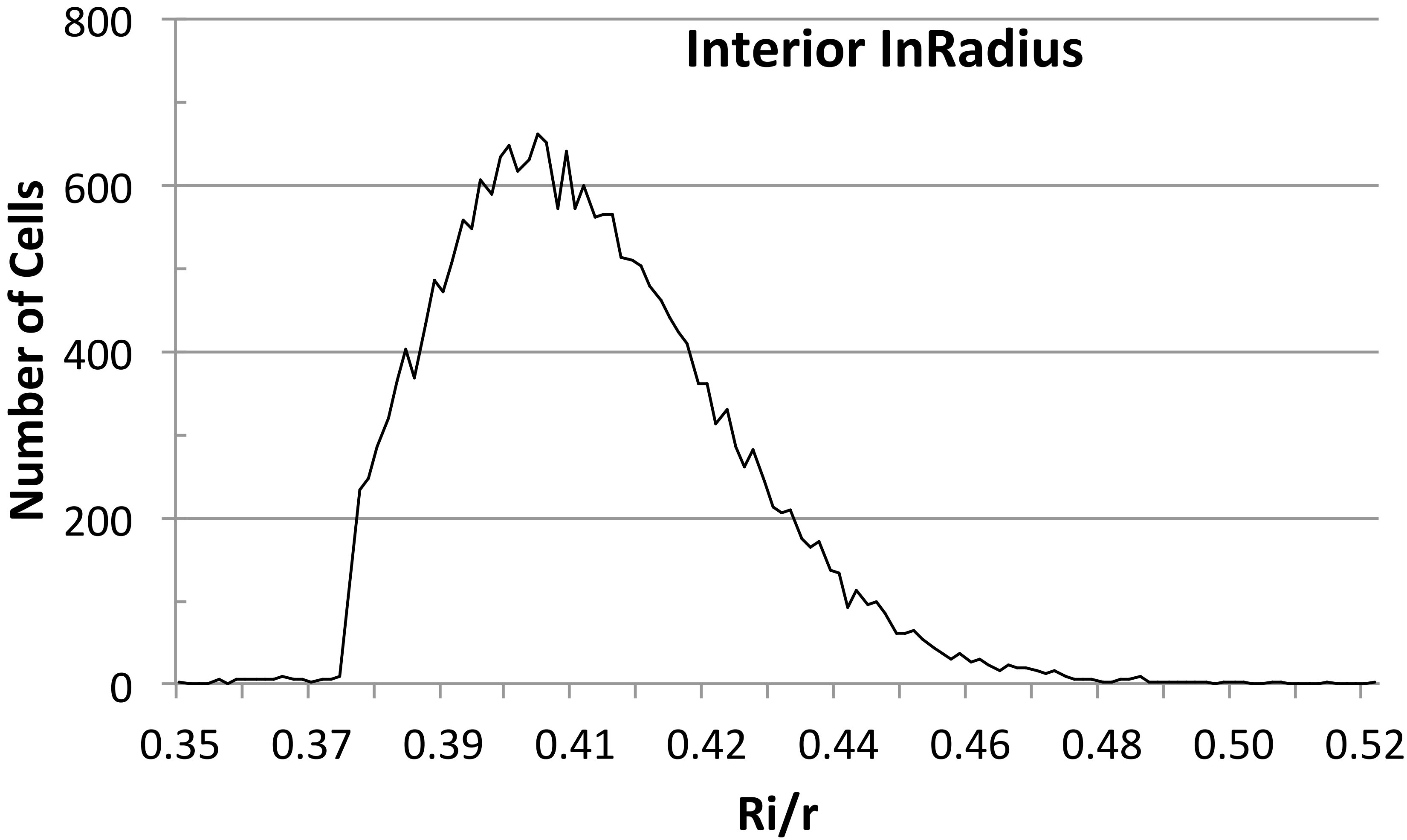


Facets of one vertex

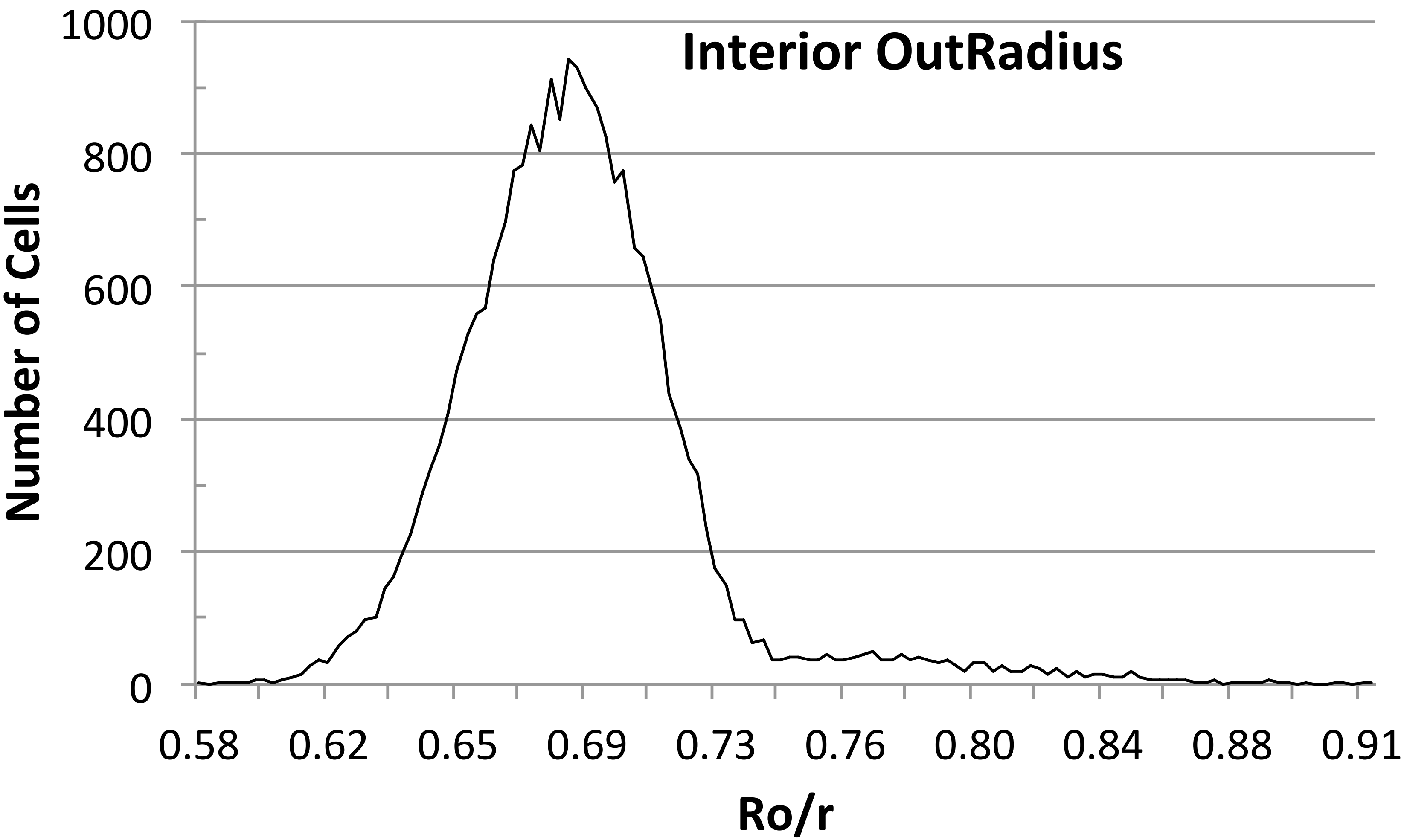
Disjoint facets: feature size *fs*

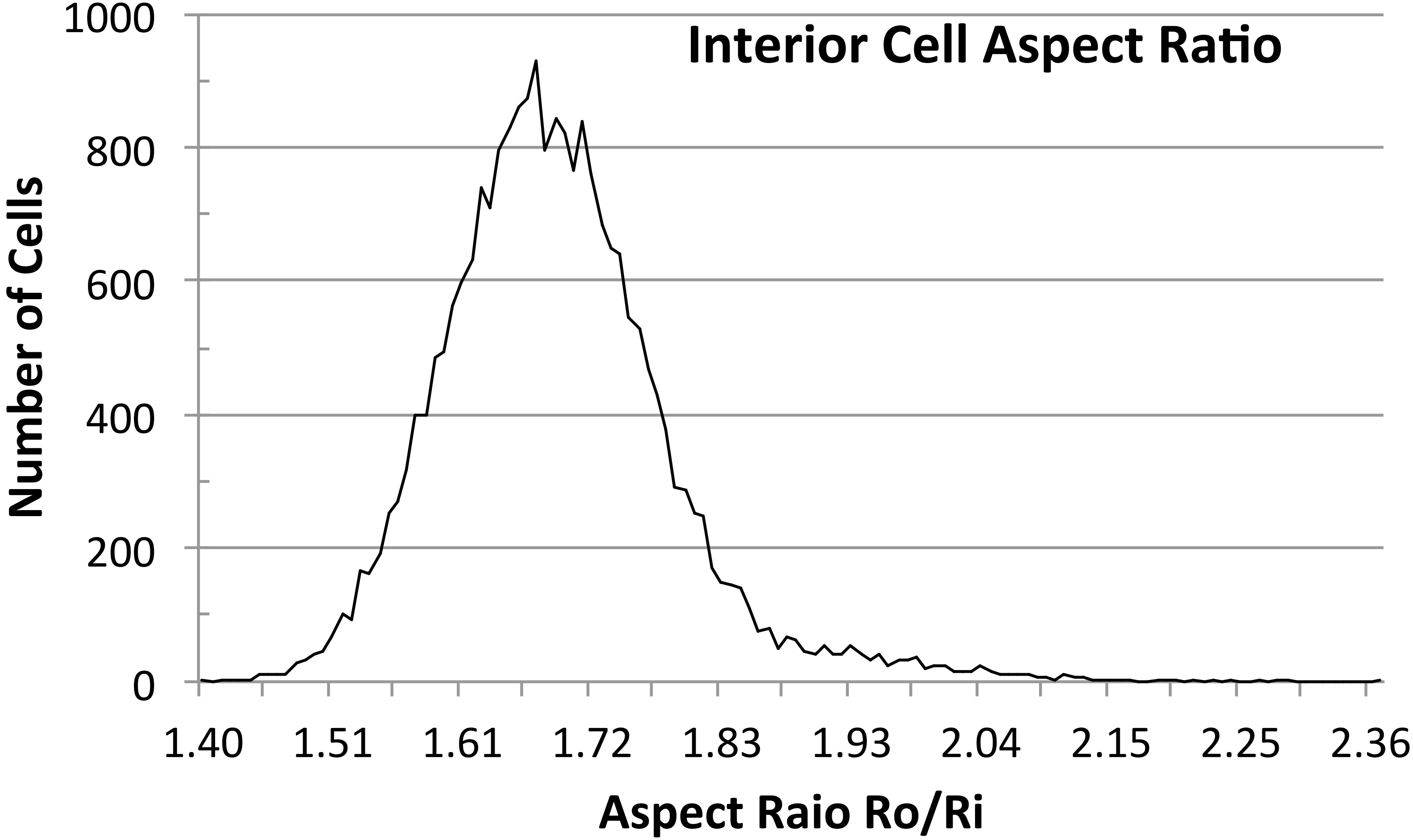
$$A \leq 4 \max\left(\sqrt{2}, r / fs\right) \max\left(1, \frac{1+\sin\omega}{2\sin\omega}\right)$$

Interior cells



Interior cells

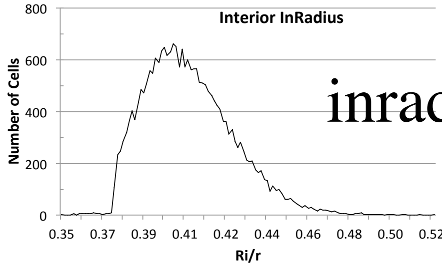




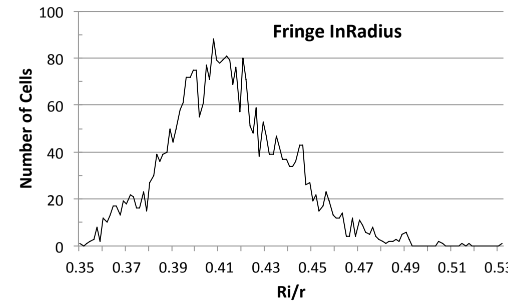
Interior Cell Aspect Ratio
Number of Cells
0
200
400
600
800
1000
1.40
1.51
1.61
1.72
1.83
1.93
2.04
2.15
2.25
2.36
Aspect Raio Ro/Ri

Observed Aspect Ratio

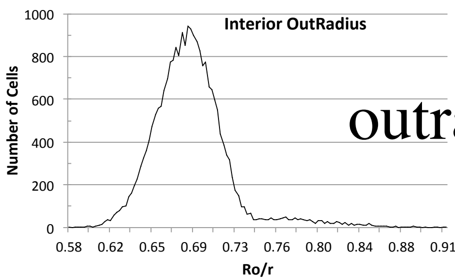
$$A \le 4\max\left(\sqrt{2}, r/fs\right)\max\left(1, \frac{1+\sin\omega}{2\sin\omega}\right)$$



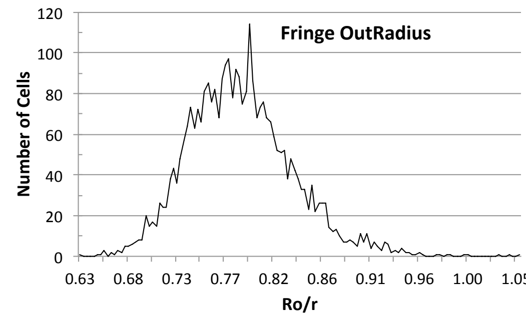
inradius

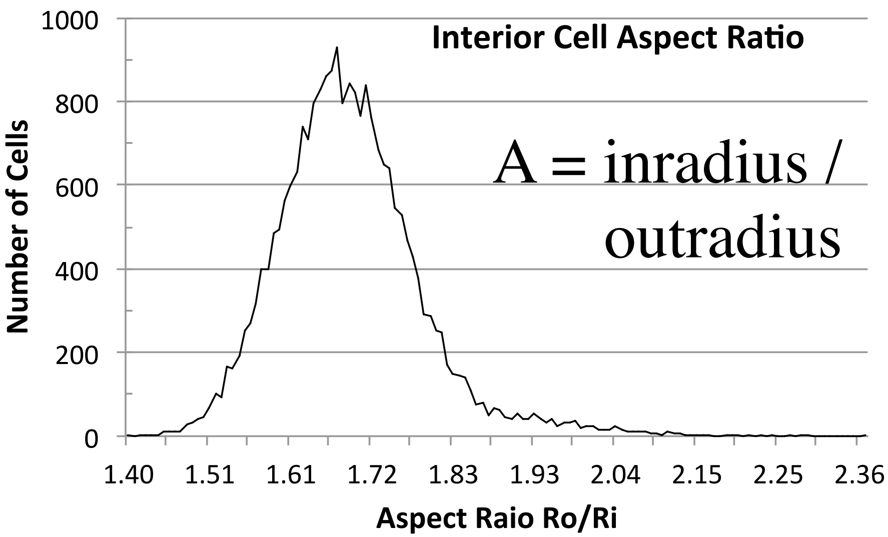


star-shaped cells

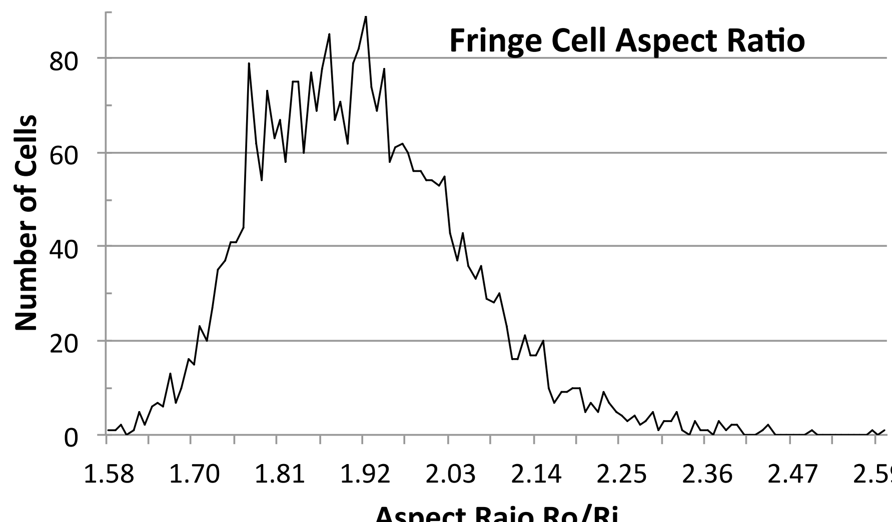


outradius

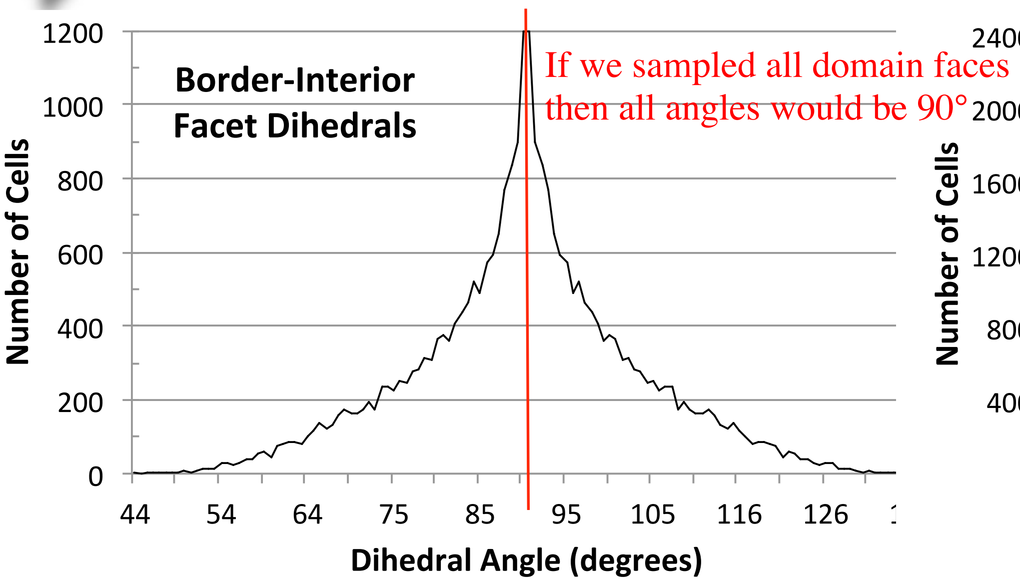


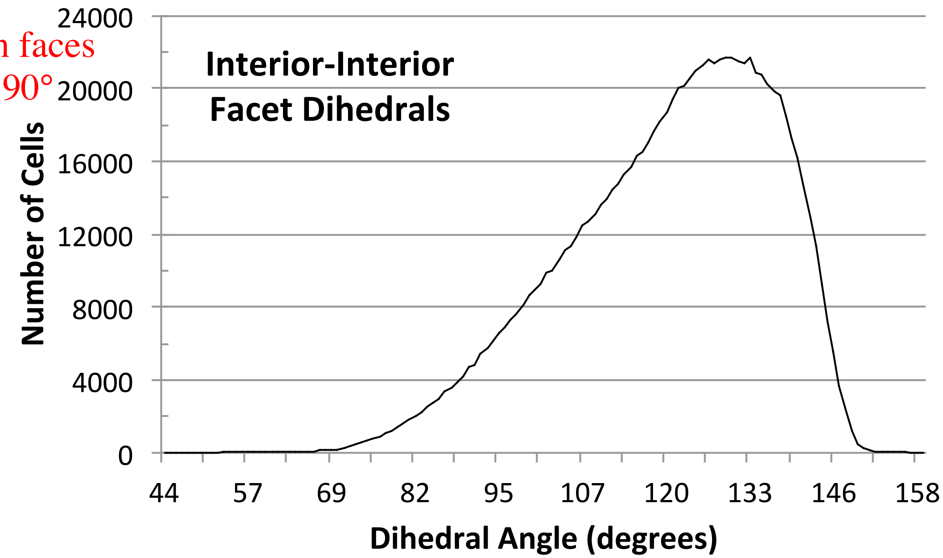


A = inradius / outradius



Quality plots
Dihedral Angles





provably $\in [30°, 150°]$ near one border facet

provably $\in [20.7°, 159.7°]$ otherwise

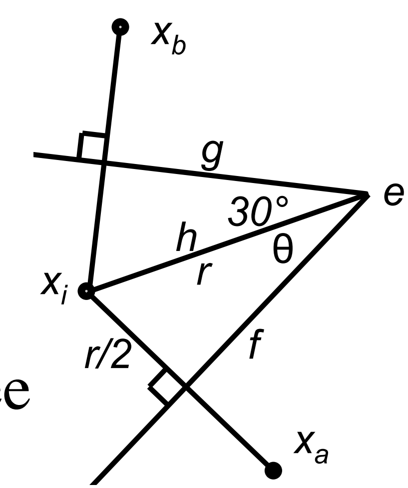
provably $\in [60°, 150°]$

Recall proofs idea:

Distance from seed to

cell vertex bounded above by maximality

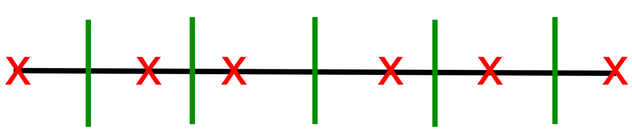
cell facet distance bounded below by disk-free

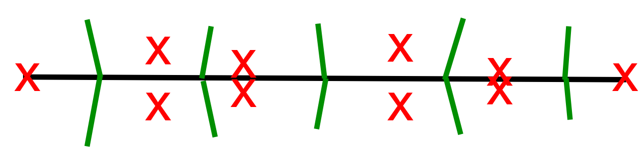


Quality: what's missing?

Work in progress:

- **Short edges**
  - **Collapsed, leading to non-planar faces**
  - **OK for Joe Bishop fracture simulation but not ideal**
- **Voronoi *facet* aspect ratio bounds**
  - **Smoothing or sample insertion constraints may fix**
- **90° facet dihedrals between samples *on reflex faces.* (Recall no samples *on* other faces)**
  - **Small random perpendicular offsets may fix**





Conclusions

- w/ Patney, Davidson, Owens (UC Davis)
- w/ Knupp, Bishop, Martinez, Leung (SNL)

- 1. Maximal Poisson-disk sampling point clouds
    - Essence: First provable maximal, bias-free, O(n) space, E(n log n) time
    - Impact: Graphics hot topic (texture synthesis). Ensemble calculations for V&V

  2. Triangular meshes
    - Essence: Provable quality bounds from random points
    - Impact: Seismic simulations

  3. Voronoi meshes
    - Essence: NOT the dual of a boundary-fitted triangulation
    - Impact: Fracture simulations

***Efficient Maximal Poisson-Disk Sampling*.**
Ebeida, Patney, Mitchell, Davidson, Knupp & Owens.
SIGGRAPH 2011. ACM Transactions on Graphics.

***Efficient and Good Delaunay Meshes From Random Points*.**
Ebeida, Mitchell, Davidson, Patney, Knupp & Owens.
SIAM Conference on Geometric and Physical Modeling.
J Computer-Aided Design special issue.

***Uniform Random Voronoi Meshes*.**
Ebeida & Mitchell.
International Meshing Roundtable, Oct 2011.

- ***Community should consider using maximal samples for mesh points****… even if Poisson-disk process isn’t important*
  - *Better sizing control.*
  - *Never $O(n^2)$*
    - *To do: study element count and grading vs. Delaunay refinement.*